

On restricted edge-connectivity of half-transitive multigraphs *

Yingzhi Tian [†] Jixiang Meng

College of Mathematics and System Sciences, Xinjiang University,
Urumqi, Xinjiang, 830046, Peoples Republic of China.

Abstract Let $G = (V, E)$ be a multigraph (it has multiple edges, but no loops). We call G maximally edge-connected if $\lambda(G) = \delta(G)$, and G super edge-connected if every minimum edge-cut is a set of edges incident with some vertex. The restricted edge-connectivity $\lambda'(G)$ of G is the minimum number of edges whose removal disconnects G into non-trivial components. If $\lambda'(G)$ achieves the upper bound of restricted edge-connectivity, then G is said to be λ' -optimal. A bipartite multigraph is said to be half-transitive if its automorphism group is transitive on the sets of its bipartition. In this paper, we will characterize maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and λ' -optimal half-transitive multigraphs.

Keywords: Multigraphs; Half-transitive multigraphs; Maximally edge-connected; Super edge-connected; Restricted edge-connectivity.

1 Introduction

A graph G consists of vertex set V and edge set E , where E is a multiset of unordered pairs of (not necessarily distinct) vertices. A *loop* is an edge whose endpoints are the same vertex. An edge is *multiple* if there is another edge with the same endvertices; otherwise it is simple. The *multiplicity* of an edge e , denoted by $\mu(e)$, is the number of multiple edges sharing the same endvertices; the *multiplicity* of a graph G , denoted by $\mu(G)$, is the maximum multiplicity of its edges. A graph is a *simple graph* if it has no multiple edges or loops, a *multigraph* if it has multiple edges, but no loops, and a *pseudograph* if it contains both multiple edges and loops. The *underlying graph* of a multigraph G ,

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[†]Corresponding author. E-mail: tianyzhxj@163.com (Y.Tian), mjjx@xju.edu.cn (J.Meng).

denoted by $U(G)$, is a simple graph obtained from G by destroying all multiple edges. It is clear that $\mu(G) = 1$ if the graph G is simple.

Let $G = (V, E)$ be a multigraph. Denote by $\lambda(G)$ the edge-connectivity of G . For $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G , a multigraph G with $\lambda(G) = \delta(G)$ is naturally said to be *maximally edge-connected*, or *λ -optimal* for simplicity. A multigraph G is said to be *vertex-transitive* if for any two vertices u and v in G , there is an automorphism α of G such that $v = \alpha(u)$, that is, $\text{Aut}(G)$ acts transitively on V . A bipartite multigraph G with bipartition $V_1 \cup V_2$ is called *half-transitive* if $\text{Aut}(G)$ acts transitively both on V_1 and V_2 . Mader [9] proved the following well-known result.

Theorem 1.1. [9] *Every connected vertex-transitive simple graph G is λ -optimal.*

If G is a vertex-transitive multigraph, then G is not always maximally edge-connected. A simple example is the multigraph obtained from a 4-cycle C_4 by replacing each edge belongs to a pair of opposite edges in C_4 with m ($m \geq 2$) multiple edges.

For half-transitive simple graphs, Liang and Meng [8] proved the following result:

Theorem 1.2. [8] *Every connected half-transitive simple graph G is λ -optimal.*

The problem of exploring edge-connected properties stronger than the maximally edge-connectivity for simple graphs has been the theme of many research. The first candidate may be the so-called *super edge-connectivity*. We can generalize this definition to multigraphs. A multigraph G is said to be *super edge-connected*, in short, *super- λ* , if each of its minimum edge-cut sets isolates a vertex, that is, every minimum edge-cut is a set of edges incident with a certain vertex in G . By the definitions, a super- λ multigraph must be a λ -optimal multigraph. However, the converse is not true. For example, $K_m \times K_2$ is λ -optimal but not super- λ since the set of edges between the two copies of K_m is a minimum edge-cut which does not isolate any vertex.

The concept of super- λ was originally introduced by Bauer et al. see [1], where combinatorial optimization problems in design of reliable probabilistic simple graphs were investigated. The following theorem is a nice result of Tindell [15], which characterized super edge-connected vertex-transitive simple graphs.

Theorem 1.3. [15] *A connected vertex-transitive simple graph G which is neither a cycle nor a complete graph is super- λ if and only if it contains no clique K_k where k is the degree of G .*

For further study, Esfahanian and Hakimi [4] introduced the concept of restricted edge-connectivity for simple graphs. The concept of restricted edge-connectivity is one kind of conditional edge-connectivity proposed by Harary in [5], and has been successfully applied in the further study of tolerance and reliability of networks, see [2,3,7,11,12,18,20-22]. Let F be a set of edges in G . Call F a *restricted edge-cut* if $G - F$ is disconnected and contains no isolated vertices. The minimum cardinality over all restricted edge-cuts

is called *restricted edge-connectivity* of G , and denoted by $\lambda'(G)$. It is shown by Wang and Li [17] that the larger $\lambda'(X)$ is, the more reliable the network is. In [4], it is proved that if a connected simple graph G of order $|V(G)| \geq 4$ is not a star $K_{1,n-1}$, then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of G . A simple graph G with $\lambda'(G) = \xi(G)$ is called a λ' -*optimal graph*. It should be pointed out that if $\delta(G) \geq 3$, then a λ' -optimal simple graph must be super- λ . In fact, a graph G is super- λ if and only if $\lambda(G) < \lambda'(G)$, see [6]. Thus, the concepts of λ -optimal graphs, super- λ graphs and λ' -optimal graphs describe reliable interconnection structures for graphs at different levels.

In [10], Meng studied the parameter λ' for connected vertex-transitive simple graphs. The main result may be restate as follows:

Theorem 1.4. [10] *Let G be a k -regular connected vertex-transitive simple graph which is neither a cycle nor a complete graph. Then G is not λ' -optimal if and only if it contains a $(k-1)$ -regular subgraph H satisfying $k \leq |V(H)| \leq 2k-3$.*

The authors in [13] proved the following result.

Theorem 1.5. [13] *Let $G = (V_1 \cup V_2, E)$ be a connected half-transitive simple graph with $n = |V(G)| \geq 4$ and $G \not\cong K_{1,n-1}$. Then G is λ' -optimal.*

Since a graph G is super- λ if and only if $\lambda(G) < \lambda'(G)$, Theorem 1.5 implies the following corollary.

Corollary 1.6. *The only connected half-transitive simple graphs which are not super- λ are cycles $C_n (n \geq 4)$.*

We can naturally generalize the concept of restricted edge-connectivity to multigraphs. The *restricted edge-connectivity* $\lambda'(G)$ of a multigraph G is the minimum number of edges whose removal disconnects G into non-trivial components. Similarly, define the minimum edge degree of G as $\xi(G) = \min\{\xi(e) = d(u) + d(v) - 2\mu(e) : e = uv \in E(G)\}$, where $\xi(e) = d(u) + d(v) - 2\mu(e)$ is the edge degree of the edge $e = uv$ in G . By using a similar argument as in [4], we can prove that the restricted edge-connectivity of a connected multigraph G is well-defined if $|V(G)| \geq 4$ and $U(G) \not\cong K_{1,n-1}$, but the inequality $\lambda'(G) \leq \xi(G)$ is not always correct. For example, the restricted edge-connectivity of the multigraph G in Fig.1 is 6, but $\xi(G) = 4$.

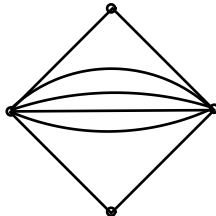


Fig.1

In [14], we gave sufficient and necessary conditions for vertex-transitive multigraphs to be maximally edge-connected, super edge-connected and λ' -optimal. In the following, we will study maximally edge-connected half-transitive multigraphs, super edge-connected half-transitive multigraphs, and λ' -optimal half-transitive multigraphs.

2 Preliminary

Let $G = (V, E)$ be a multigraph. For two disjoint non-empty subsets A and B of V , let $[A, B] = \{e = uv \in E : u \in A \text{ and } v \in B\}$. For the sake of convenience, we write u for the single vertex set $\{u\}$. If $\bar{A} = V \setminus A$, then we write $N(A)$ for $[A, \bar{A}]$ and $d(A)$ for $|N(A)|$. Thus $d(u)$ is just the degree of u in G . Denote by $G[A]$ the subgraph of G induced by A .

An edge-cut F of G is called a λ -cut if $|F| = \lambda(G)$. It is easy to see that for any λ -cut F , $G - F$ has exactly two components. If $N(A)$ is a λ -cut of G , then A is called a λ -fragment of G . It is clear that if A is a λ -fragment of G , then so is \bar{A} . Let $r(G) = \min\{|A| : A \text{ is a } \lambda\text{-fragment of } G\}$. Obviously, $1 \leq r(G) \leq \frac{1}{2}|V|$. A λ -fragment B is called a λ -atom of G if $|B| = r(G)$. A λ -fragment C is called a *strict λ -fragment* if $2 \leq |C| \leq |V(G)| - 2$. If G contains strict λ -fragments, then the ones with smallest cardinality are called λ -superatoms.

Similarly, we can give the definition of λ' -atom. A restricted edge-cut F of G is called a λ' -cut if $|F| = \lambda'(G)$. For any λ' -cut F , $G - F$ has exactly two components. Let A be a proper subset of V . If $N(A)$ is a λ' -cut of G , then A is called a λ' -fragment of G . It is clear that if A is a λ' -fragment of G , then so is \bar{A} . Let $r'(G) = \min\{|A| : A \text{ is a } \lambda'\text{-fragment of } G\}$. Obviously, $2 \leq r'(G) \leq \frac{1}{2}|V|$. A λ' -fragment B is called a λ' -atom of G if $|B| = r'(G)$.

For a multigraph G , the inequality $\lambda'(G) \leq \xi(G)$ is not always correct. But if G is a k -regular multigraph, we proved the following result.

Lemma 2.1. [14] *Let G be a connected k -regular multigraph. Then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$ if $|V(G)| \geq 4$.*

We call a bipartite multigraph G with bipartition $V_1 \cup V_2$ *semi-regular* if each vertex in V_1 has the same degree d_1 and each vertex in V_2 has the same degree d_2 . For semi-regular bipartite multigraphs, a similar result can be obtained.

Lemma 2.2. *Let G be a connected semi-regular bipartite multigraph with bipartition $V_1 \cup V_2$. Then $\lambda'(G)$ is well-defined and $\lambda'(G) \leq \xi(G)$ if $|V(G)| \geq 4$ and $U(G) \not\cong K_{1,n-1}$.*

Proof. Assume each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2 . Assume, without loss of generality, that $d_1 \leq d_2$. Let $e = uv$ be an edge such that $\xi(e) = \xi(G)$, where $u \in V_1$ and $v \in V_2$. If $G - \{u, v\}$ contains a non-trivial component, say C , then $N(V(C))$ is a restricted edge-cut and $|N(V(C))| \leq |N(\{u, v\})| = \xi(e) = \xi(G)$.

Thus assume that $G - \{u, v\}$ only contains isolated vertices. If there is a vertex w other than v in V_2 , then $d_1 + d_2 \leq |N(V \setminus \{u, v\})| = |N(\{u, v\})| = \xi(e) = d_1 + d_2 - \mu(e) < d_1 + d_2$ by $|V(G)| \geq 4$, a contradiction. Thus $V_2 = \{v\}$ and $U(G) \cong K_{1, n-1}$, also a contradiction. \square

Because of Lemma 2.1 and Lemma 2.2, we call a regular multigraph (or a semi-regular bipartite multigraph) G λ' -optimal if $\lambda'(G) = \xi(G)$. Since each vertex-transitive multigraph is regular and each half-transitive multigraph is semi-regular, thus a vertex-transitive multigraph (or a half-transitive multigraph) G is λ' -optimal if $\lambda'(G) = \xi(G)$.

Recall that an *imprimitive block* for a permutation group Φ on a set T is a proper, non-trivial subset A of T such that for every $\varphi \in \Phi$ either $\varphi(A) = A$ or $\varphi(A) \cap A = \emptyset$. A subset A of $V(G)$ is called an *imprimitive block* for G if it is an imprimitive block for the automorphism group $\text{Aut}(G)$ on $V(G)$. The following theorem shows the importance of imprimitive blocks:

Theorem 2.3. [16] *Let $G = (V, E)$ be a connected simple graph and A be an imprimitive block for G . If G is vertex-transitive, then $G[A]$ is also vertex-transitive.*

By a similar argument as Theorem 2.3, we can obtain the following result for half-transitive multigraphs.

Lemma 2.4. *Let G be a connected bipartite multigraph with bipartition $V_1 \cup V_2$. Assume A is an imprimitive block for G such that $A \cap V_1 \neq \emptyset$ and $A \cap V_2 \neq \emptyset$. If G is half-transitive, then $G[A]$ is also half-transitive.*

Proof. Since G is half-transitive, for any two vertices $u, v \in A \cap V_i$ ($i \in \{1, 2\}$), there is $\alpha \in \text{Aut}(G)$ such that $\alpha(u) = v$. Because $\alpha(A) \cap A \neq \emptyset$, we have $\alpha(A) = A$ by A is an imprimitive block for G . Thus the restriction of α to A is an automorphism of $G[A]$, which maps u to v . It follows $G[A]$ is a half-transitive multigraph.

3 Maximally edge-connected half-transitive multigraphs

In [9], Mader proved that any two distinct λ -atoms of a simple graph are disjoint. For multigraphs, this property still holds.

Lemma 3.1. *Let G be a connected multigraph. Then any two distinct λ -atoms of G are disjoint.*

Proof. Suppose to the contrary that there are two distinct λ -atoms A and B with $A \cap B \neq \emptyset$. We have $V(G) \setminus (A \cup B) \neq \emptyset$ by $|A| \leq |V(G)|/2$ and $|B| \leq |V(G)|/2$. Then $N(A \cap B)$ and $N(A \cup B)$ are edge-cuts of G , thus $d(A \cap B) = |N(A \cap B)| \geq \lambda(G)$ and $d(A \cup B) = |N(A \cup B)| \geq \lambda(G)$. From the following well-known submodular inequality (see [16]),

$$2\lambda(G) \leq d(A \cup B) + d(A \cap B) \leq d(A) + d(B) = 2\lambda(G),$$

we conclude that both $|d(A \cap B)| = \lambda(G)$ and $|d(A \cup B)| = \lambda(G)$ hold. Thus $A \cap B$ is a λ -fragment with $|A \cap B| < |A|$, which contradicts to A is a λ -atom of G . \square

Theorem 3.2. *Let G be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$. Assume each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2 . Then G is not maximally edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph H of G such that*

$$|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1,$$

where $A_1 = V_1 \cap V(H)$, $A_2 = V_2 \cap V(H)$, d'_1 is the degree of each vertex of A_1 in H and d'_2 is the degree of each vertex of A_2 in H .

Proof. Assume, without loss of generality, that $d_1 \leq d_2$. If G is not maximally edge-connected, then $\lambda(G) \leq d_1 - 1$. Let A be a λ -atom of G and $H = G[A]$. By Lemma 3.1, we know A is an imprimitive block for G . Thus H is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in $A \cap V_1$ has degree d'_1 in H and each vertex in $A \cap V_2$ has degree d'_2 in H . Then $|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda(G) \leq d_1 - 1$.

Now we prove the sufficiency. Assume G contains a proper induced connected half-transitive multi-subgraph H such that $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1$, then $\lambda(G) \leq d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\} - 1$, that is, G is not maximally edge-connected. \square

4 Super edge-connected half-transitive multigraphs

In [16], Tindell studied the intersection property of λ -superatoms of vertex-transitive simple graphs. For half-transitive multigraphs, we have the following lemma.

Lemma 4.1. *Let G be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$. Assume G is not super edge-connected, A and B are two distinct λ -superatoms. If $|A| = |B| \geq 3$, then $A \cap B = \emptyset$.*

Proof. Assume each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2 . Without loss of generality, assume that $d_1 \leq d_2$. If $A \cap B \neq \emptyset$, then by a similar argument as the proof of Lemma 3.1, we can conclude that $|d(A \cap B)| = |d(A \cup B)| = \lambda(G)$. We claim that $|A \cap B| = 1$. Otherwise, if $|A \cap B| \geq 2$, then $|V(G) \setminus (A \cup B)| \geq |A \cap B| \geq 2$. Since $G[A]$, $G[V \setminus A]$, $G[B]$ and $G[V \setminus B]$ are connected, we have $G[A \cup B]$ and $G[V \setminus (A \cap B)]$ are connected. If $G[A \cap B]$ is not connected, then we have $d(A \cap B) \geq 2\lambda(G)$, a contradiction. If $G[A \cap B]$ is connected, then $A \cap B$ is a strict λ -fragment with $|A \cap B| < |A|$, which contradicts to A is a λ -superatom. Hence $|A \cap B| = 1$.

Let $C = V(G) \setminus B$. Then $|A \cap C| = |A \setminus (A \cap B)| \geq 2$, and A , $V(G) \setminus A$, C and $V(G) \setminus C$ are all strict λ -fragments. By a similar argument as above we can deduce that $A \cap C$ is a strict λ -fragment with $|A \cap C| < |A|$, which is impossible. \square

Theorem 4.2. *Let G be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$. Assume each vertex in V_1 has degree d_1 , each vertex in V_2 has degree d_2 and $|V(G)| \geq 2 \min\{d_1, d_2\} + 2$. Then G is not super edge-connected if and only if there is a proper induced connected half-transitive multi-subgraph H of G such that*

$$|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\},$$

where $A_1 = V_1 \cap V(H)$, $A_2 = V_2 \cap V(H)$, d'_1 is the degree of each vertex of A_1 in H and d'_2 is the degree of each vertex of A_2 in H .

Proof. Assume, without loss of generality, that $d_1 \leq d_2$. If G is not super edge-connected, then G contains λ -superatoms. Let A be a λ -superatom of G and $H = G[A]$. If $|A| = 2$, then H is isomorphic to a multigraph which contains two vertices and t edges between these two vertices. Thus H is an induced t -regular connected half-transitive multi-subgraph of G . Therefore $|A \cap V_1|(d_1 - t) + |A \cap V_2|(d_2 - t) = d(A) = \lambda(G) \leq d_1$. In the following, we assume that $|A| \geq 3$.

By Lemma 4.1, we know A is an imprimitive block for G . Thus H is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in $A \cap V_1$ has degree d'_1 in H and each vertex in $A \cap V_2$ has degree d'_2 in H . Thus $|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda(G) \leq d_1$.

Now we prove the sufficiency. Assume G contains a proper induced connected half-transitive multi-subgraph H such that $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$, then $d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$. If $G - V(H)$ contains no isolated vertices, then $V(H)$ is a strict λ -fragment. Thus G is not super edge-connected. Assume $G - V(H)$ contains an isolated vertex w , then $N(w) = N(V(H))$. Since $|A_1| \leq \min\{d_1, d_2\}$ and $|A_2| \leq \min\{d_1, d_2\}$ by $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \min\{d_1, d_2\}$, we see that G is not connected by $|V(G)| \geq 2 \min\{d_1, d_2\} + 2$, a contradiction. \square

5 λ' -optimal half-transitive multigraphs

In [19], the authors proved the following fundamental result for studying the restricted edge-connectivity of simple graphs.

Theorem 5.1. [19] *Let $G = (V, E)$ be a connected simple graph with at least four vertices and $G \not\cong K_{1,n-1}$. If G is not λ' -optimal, then any two distinct λ' -atoms of G are disjoint.*

For multigraphs, we cannot obtain a similar result as in Theorem 5.1. But for half-transitive multigraphs, the similar result holds.

Lemma 5.2. *Let G be a connected multigraph with $\delta(G) \geq 2\mu(G)$. If G contains a λ' -atom A with $|A| \geq 3$, then each vertex in A has at least two neighbors in A .*

Proof. By contradiction, assume there is a vertex $u \in A$ such that u contains only one neighbor in A . Let v be the only neighbor of u in A . Set $A' = A \setminus \{u\}$. Then both $G[A']$ and $G[\overline{A'}]$ are connected. We have $|A'| \geq 2$ by $|A| \geq 3$. Clearly, $|\overline{A'}| = |\overline{A}| + 1 \geq 4$. Thus $[A', \overline{A'}]$ is a restricted edge-cut. Since $\delta(G) \geq 2\mu(G)$, we have

$$\lambda'(G) \leq |[A', \overline{A'}]| = |[A, \overline{A}]| + \mu(uv) - (d(u) - \mu(uv)) \leq |[A, \overline{A}]| = \lambda'(G).$$

It follows that A' is a λ' -fragment with $|A'| < |A|$, which contradicts to A is a λ' -atom. \square

Lemma 5.3. *Let G be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$ and $\delta(G) \geq 2\mu(G)$. Assume G is not λ' -optimal, A and B are two distinct λ' -atoms. Then $|A| = |B| \geq 3$ and $A \cap B = \emptyset$.*

Proof. Assume each vertex in V_1 has degree d_1 and each vertex in V_2 has degree d_2 . Without loss of generality, assume that $d_1 \leq d_2$.

If $|A| = 2$, then $\lambda'(G) = d(A) = d_1 + d_2 - 2\mu(uv) \geq \xi(G)$ (where $A = \{u, v\}$), which contradicts that G is not λ' -optimal. Thus $|A| \geq 3$.

Suppose to the contrary that $A \cap B \neq \emptyset$. Set $C = A \cap B$, $A_1 = A \cap \overline{B}$, $B_1 = B \cap \overline{A}$ and $D = \overline{A} \cap \overline{B} = \overline{A} \cup \overline{B}$. In the following, we will derive a contradiction by a series of claims.

Clearly, one of the following two inequalities must holds:

$$|[A_1, C]| \leq |[C, B_1]| + |[C, D]|, \quad (1)$$

$$|[B_1, C]| \leq |[C, A_1]| + |[C, D]|. \quad (2)$$

In the following, we always assume, without loss of generality, that inequality (1) holds.

Claim 1. A_1 satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}(v_{21} \in V_2)$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \dots, v_{1m}\}(v_{1i} \in V_1 \text{ for } 1 \leq i \leq m)$ and $d_2 > (m-1)d_1 + 2\mu(G)$.

It follows from inequality (1) that

$$d(A_1) = |[A_1, D]| + |[A_1, C]| + |[A_1, B_1]| \leq d(A) = \lambda'(X).$$

Assume $G[A_1]$ has a component \tilde{G} with $|V(\tilde{G})| \geq 2$. Set $F = V(\tilde{G})$. Since $G[B]$ and $G[\overline{A}]$ are both connected, and $B \cap \overline{A} \neq \emptyset$, we see that $G[\overline{A_1}]$ is connected. Furthermore, since G is connected, every component of $G[A_1]$ is joined to $G[\overline{A_1}]$, and thus $G[\overline{F}]$ is connected. So $[F, \overline{F}]$ is a restricted edge-cut with $|d(F)| \leq \lambda'(G)$. Because A is a λ' -atom and F is a proper subset of A , we obtain $d(F) > d(A) = \lambda'(G)$, it is a contradiction. Thus, each component in $G[A_1]$ is an isolated vertex. By $d(A_1) \leq \lambda'(G) < d_1 + d_2 - 2\mu(G)$, we can derive that A_1 satisfies one of the following two conditions: (i) $A_1 = \{v_{21}\}(v_{21} \in V_2)$ and $d_1 > 2\mu(G)$, or (ii) $A_1 = \{v_{11}, \dots, v_{1m}\}(v_{1i} \in V_1 \text{ for } 1 \leq i \leq m)$ and $d_2 > (m-1)d_1 + 2\mu(G)$.

Claim 2. $C \not\subseteq V_1$ and $C \not\subseteq V_2$.

By contradiction. Suppose $C \subseteq V_1$. Then $G[C]$ is an independent set. Since we have assumed that $||[A_1, C]|| \leq ||[C, B_1]|| + ||[C, D]||$, there exists a vertex v in C such that

$$|[v, A_1]| \leq |[v, D]| + |[v, B_1]|. \quad (3)$$

Set $F = A \setminus \{v\}$, then

$$d(F) = d(A) - |[v, D]| - |[v, B_1]| + |[v, A_1]| \leq d(A) = \lambda'(X).$$

Since $G[A]$ is connected and C is an independent set, we have $|[v, A_1]| \geq 1$. It follows from inequality (3) that $|[v, \overline{A}]| \geq 1$. So, $G[\overline{F}]$ is connected. We claim that each component in $G[F]$ has at least 2 vertices. In fact, if there is an isolated vertex u in $G[F]$, then v is the only vertex adjacent to u in $G[A]$, which contradicts to Lemma 5.2. Now, similarly as in the proof of Claim 1, a contradiction arises, since F contains a smaller λ' -fragment than A . $C \not\subseteq V_2$ can be proved similarly.

Claim 3. $d(D) < \lambda'(G)$ and D is an independent set contained in V_1 .

By Claim 2, $|C| \geq 2$. We claim that $d(C) > \lambda'(G)$. In fact, if $G[C]$ contains a component of order at least 2, then similar to the proof of Claim 1, we can show that $[C, \overline{C}]$ contains a restricted edge-cut, and thus $d(C) > \lambda'(G)$. Otherwise, we assume that each component in $G[C]$ is an isolated vertex. Since not all vertices in C are from the same bipartition, there must be at least one vertex in V_2 . From $|C| \geq 2$, we have $d(C) \geq d_2 + d_1 > \xi(G) \geq \lambda'(X)$. Thus, we have that $d(C) > \lambda'(G)$.

From the well-known submodular inequality (see [16]), we have

$$d(C) + d(D) \leq d(A) + d(B) = 2\lambda'(G). \quad (4)$$

By (4) and $d(C) > \lambda'(G)$, we obtain $d(D) < \lambda'(G)$. Applying a similar argument as above, we can show that D is an independent set contained in V_1 .

Let $s = |D|$. Then $s \geq 2$ and

$$d(D) = sd_1. \quad (5)$$

Denote by e_1 the number of edges in $G[\overline{C}]$. Clearly,

$$d(C) = d(\overline{C}) = \sum_{v \in \overline{C}} d(v) - 2e_1. \quad (6)$$

Since $G[\overline{B}]$ is connected and D is an independent set contained in V_1 , Claim 1 (ii) can not hold. Thus, Claim 1 (i) is true. Since G is a bipartite multigraph, we have

$$e_1 \leq 2s\mu(G). \quad (7)$$

Combining this with (4), (5) and (6), we see that

$$2d_1 + 2d_2 - 4\mu(G) - sd_1 > 2\lambda'(G) - d(D) \geq d(C) \geq sd_1 + 2d_2 - 4s\mu(G).$$

This implies $d_1 < 2\mu(G)$, contradicting to the assumption that $d_1 \geq 2\mu(G)$. \square

Theorem 5.4. *Let G be a connected half-transitive multigraph with bipartition $V_1 \cup V_2$ and $\delta(G) \geq 2\mu(G)$. Assume each vertex in V_1 has degree d_1 , each vertex in V_2 has degree d_2 , $|V_1| \geq \xi(G)$ and $|V_2| \geq \xi(G)$. Then G is not λ' -optimal if and only if there is a proper induced connected half-transitive multi-subgraph H of G such that*

$$|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1,$$

where $A_1 = V_1 \cap V(H)$, $A_2 = V_2 \cap V(H)$, d'_1 is the degree of each vertex of A_1 in H and d'_2 is the degree of each vertex of A_2 in H .

Proof. Assume, without loss of generality, that $d_1 \leq d_2$. If G is not λ' -optimal, then G contains λ' -atoms. Let A be a λ' -atom of G and $H = G[A]$. By Lemma 5.3, we have $|A| \geq 3$ and A is an imprimitive block for G . Thus H is a connected half-transitive multigraph by Lemma 2.4. Assume each vertex in $A \cap V_1$ has degree d'_1 in H and each vertex in $A \cap V_2$ has degree d'_2 in H . Thus $|A \cap V_1|(d_1 - d'_1) + |A \cap V_2|(d_2 - d'_2) = d(A) = \lambda'(G) \leq \xi(G) - 1$.

Now we prove the sufficiency. Assume G contains a proper induced connected half-transitive multi-subgraph H such that $|A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1$, then $d(V(H)) = |A_1|(d_1 - d'_1) + |A_2|(d_2 - d'_2) \leq \xi(G) - 1$, $|A_1| \leq \xi(G) - 1$ and $|A_2| \leq \xi(G) - 1$. If $G - V(H)$ contains a non-trivial component, say B , then $[B, \overline{B}]$ is a restricted edge-cut and $d(B) \leq d(V(H)) \leq \xi(G) - 1$. Thus G is not λ' -optimal. Now we assume that each component of $G - V(H)$ is an isolated vertex, then $|N(V(G) \setminus V(H))| \geq d_1 + d_2 > \xi(G)$ by $|V_1| \geq \xi(G)$ and $|V_2| \geq \xi(G)$. On the other hand, $|N(V(G) \setminus V(H))| = |N(V(H))| \leq \xi(G) - 1$, it is a contradiction. \square

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